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We achieved a serious progress in numberical simulation of surface waves on free surface of 3D deep fluid in presence of gravity and surface tension. We obtained important results in the theory of 2D fluid with a free surface. We performed a numerical calculation of the Kolmogorov constant in the theory of turbulence of ideal fulid in one-loop approximation in Clebsh variables.

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SEA-AIR EXCHANGE OF ENERGY AND MOMENTS UNDER WELL-DEVELOPED SEA CONDITIONS: THEORY AND EXPERIMENT

V.E. ZAKHAROV, P.I. AND E. KUZNETSOV, CO-PI

Ocotober 1, 1994-September 30, 1995

Our work in 1995 is going very succesfully. We received following essential results:

1.We achieved a serious progress in numerical simulation of surface waves on free surface of 3D deep fluid in presence of gravity and surface tension. At the moment,using a kind of spectral code,we can perform simulation of 256 x 256 spectral modes. In this problem we use expansion of the Hamiltonian up to terms of fourth order in canonical variables. The developed technique allows to solve a lot of different physical problems. So far we used it for solving a problem of a big theoretical importance. By direct simulation we comfirmed existence and stability of the weak-turbulent Kolmogorov spectrum for capilliary waves. The result is submitted for publication to the "Physical Review Letters" (Pushkarev, Zakharov).

2.We obtained important results in the theory of 2D fluid with a free surface. Using conformal mapping to half plane directly in the principle of Hamilton, we found a way of decsription of the surface which is very suitable for numerical simulation by spectral code. The simulation can be done in exact equations without any assumpions on a level of nonlinearity. The developed numerical scheme allows to describe strongly nonlinear prosesses like wave-breaking. By separating of scales we found an exactly integrable model in the the theory of deep fluid with free surface. The model can be applied to analytical decsription of the wave breaking and to the theory of nonlinaer phase of the Raileigh-Taylor instability. The results were reported on the International Symposium on Applied Mathematics in honor of 70 year of M.Kruscal(Boulder, Colorado,August 1995) (Dyachenko, Zakharov).

3.We perfomed a numerical calculation of the Kolmogorov constant in the theory of turbulence of ideal fluid in one-loop approximation in Clebsh variables. The article is submitted to the "Physica D" (Balk, Pushkarev, Zakharov).

Calculation of Kolmogorov Constant for Hydrodynamic Turbulence

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Abstract

The second method of calculation of Kolmogorov constant is based on the calculation of the analytical expression for matrix element over eight angles in K-space. Once this expression for averaged matrix element $S(k, k_1, k_2, k_3)$ is known, the problem of calculation of the constant is reduced principally to the numerical calculation of the integral containing function $S(k, k_1, k_2, k_3)$ as a kerr over some curvilinear two-dimensional domain.

1 Calculation of the collision integral on the scaling spectrum

Consider four-wave kinetic equation

$$\frac{\partial n_{\vec{k}}}{\partial t} = 4\pi \int |T_{\vec{k}\vec{k}_1\vec{k}_2\vec{k}_3}|^2 \delta(\vec{k}+\vec{k}_1+\vec{k}_2+\vec{k}_3)\delta(\omega+\omega_1-\omega_2-\omega_3)$$
$$n_{\vec{k}} n_{\vec{k}_1} n_{\vec{k}_2} n_{\vec{k}_3} \left(\frac{1}{n_{\vec{k}}}+\frac{1}{n_{\vec{k}_1}}-\frac{1}{n_{\vec{k}_2}}-\frac{1}{n_{\vec{k}_3}}\right) d\vec{k}_1 d\vec{k}_2 d\vec{k}_3 \qquad (1)$$

We are introducing the spherical system of coordinates in $\vec{K}$-space, choosing one of the axises directed along one of the diagonal of fourangle $\vec{k}, \vec{k}_1, \vec{k}_2, \vec{k}_3$ (see Fig.1) :

$$\begin{aligned} \vec{k} &= (k\cos\theta, k\sin\theta\cos\phi, k\sin\theta\sin\phi) \\ \vec{k}_i &= (k_i\cos\theta_i, k_i\sin\theta_i\cos\phi_i, k_i\sin\theta_i\sin\phi_i), \;\; i=\overline{1,3} \end{aligned}$$

We are assuming further that spectrum $n_{\vec{k}}$ and dispersion $\omega_{\vec{k}}$ are invariant with respect to rotations in $\vec{K}$-space. After averaging over four angles θ and four angles ϕ kinetic equation (1) takes the form:

$$\frac{\partial n_k}{\partial t} = \frac{1}{k^{d-1}\frac{dk}{d\omega}} \int S(k,k_1,k_2,k_3)\frac{dk}{d\omega}\frac{dk_1}{d\omega_1}\frac{dk_2}{d\omega_2}\frac{dk_3}{d\omega_3} n_k n_{k_1} n_{k_2} n_{k_3}$$
$$\left(\frac{1}{n_k}+\frac{1}{n_{k_1}}-\frac{1}{n_{k_2}}-\frac{1}{n_{k_3}}\right)\delta(\omega+\omega_1-\omega_2-\omega_3)d\omega_1 d\omega_2 d\omega_3$$

Here the function $S(k,k_1,k_2,k_3)$ is the result of averaging of matrix element over angles θ and ϕ in $\vec{K}$-space

$$S(k,k_1,k_2,k_3) =< |T_{\vec{k}\vec{k}_1\vec{k}_2\vec{k}_3}|^2 \delta(\vec{k}+\vec{k}_1+\vec{k}_2+\vec{k}_3) > (kk_1k_2k_3)^{d-1} =$$
$$= \int |T_{\vec{k}\vec{k}_1\vec{k}_2\vec{k}_3}|^2 \delta(\vec{k}+\vec{k}_1+\vec{k}_2+\vec{k}_3)\sin\theta\sin\theta_1\sin\theta_2\sin\theta_3 d\theta d\theta_1 d\theta_2 d\theta_3 \times$$
$$d\phi d\phi_1 d\phi_2 d\phi_3 (kk_1k_2k_3)^{d-1}$$

Brackets <> mean an averaging over $\vec{K}$-space angles.The domain of integration here is $[0, 2\pi]$ for angles θ and $[-\frac{\pi}{2}, \frac{\pi}{2}]$ for angles ϕ.

The frequency and scaling spectrum are defined as

$$\omega_k = C(\nu) k^{\alpha}$$

$$n_k = R k^{-\nu} = A(\nu) \omega_k^{-x} \tag{2}$$

where the functions $C(\nu)$ and $A(\nu)$ are defined by (...) and (...) correspondingly. The expressions for them are, however, unimportant for our purposes, because, as we will see, they drop out from the final expression for the Kolmogorov constant.

We will mention here only the relations following from our definitions:

$$\alpha x = \nu$$
$$R = C^{-x} A(\nu)$$

On scaling spectrum (2) the kinetic equation becomes

$$\frac{\partial n_k}{\partial t} = \left(\frac{A}{\alpha C}\right)^3 k^{\alpha - d} \int S(k, k_1, k_2, k_3) (\omega_k \omega_{k_1} \omega_{k_2} \omega_{k_3})^{-x}$$
$$\left(\omega_k^x + \omega_{k_1}^x - \omega_{k_2}^x - \omega_{k_3}^x\right) \delta(\omega + \omega_1 - \omega_2 - \omega_3) d\omega_1 d\omega_2 d\omega_3$$

Function $S(k, k_1, k_2, k_3)$ is a homogeneous one with respect to variables ω:

$$S((\epsilon\omega)^{\frac{1}{\alpha}}, (\epsilon\omega_{k_1})^{\frac{1}{\alpha}}, (\epsilon\omega_{k_2})^{\frac{1}{\alpha}}, (\epsilon\omega_{k_3})^{\frac{1}{\alpha}}) = \epsilon^{\gamma} S(\omega^{\frac{1}{\alpha}}, \omega_{k_1}^{\frac{1}{\alpha}}, \omega_{k_2}^{\frac{1}{\alpha}}, \omega_{k_3}^{\frac{1}{\alpha}})$$

where the power of homogeneouty is

$$\gamma = \frac{2\beta + 3d}{\alpha} - 1$$

Using the property of gomogeneouty of the function S one can "pull-out" the coefficient C out of the integration sign:

$$\frac{\partial n_k}{\partial t} = \left(\frac{A}{\alpha}\right)^3 \frac{\omega_k^{1-\frac{d}{\alpha}}}{C^{2\frac{d+\beta}{\alpha}}} \int S(\omega_k^{\frac{1}{\alpha}}, \omega_{k_1}^{\frac{1}{\alpha}}, \omega_{k_2}^{\frac{1}{\alpha}}, \omega_{k_3}^{\frac{1}{\alpha}})(\omega_k \omega_{k_1} \omega_{k_2} \omega_{k_3})^{-x}$$
$$\left(\omega_k^x + \omega_{k_1}^x - \omega_{k_2}^x - \omega_{k_3}^x\right) \delta(\omega + \omega_1 - \omega_2 - \omega_3) d\omega_1 d\omega_2 d\omega_3$$

Here the integration is carried out over infinite domain (see.Fig.2), which we will break into four domains I,II,III,IV. Making conformal mappings of domains II,III,IV into domain I (see [1]) we get

$$\frac{\partial n_k}{\partial t} = \left(\frac{A}{\alpha}\right)^3 \frac{\omega_k^{1-\frac{d}{\alpha}}}{C^{2\frac{d+\beta}{\alpha}}} \int S(\omega_k^{\frac{1}{\alpha}}, \omega_{k_1}^{\frac{1}{\alpha}}, \omega_{k_2}^{\frac{1}{\alpha}}, \omega_{k_3}^{\frac{1}{\alpha}})(\omega_k \omega_{k_1} \omega_{k_2} \omega_{k_3})^{-x} \times \quad (3)$$
$$\left(\omega_k^x + \omega_{k_1}^x - \omega_{k_2}^x - \omega_{k_3}^x\right) \left(1 + \left(\frac{\omega_{k_1}}{\omega_k}\right)^y - \left(\frac{\omega_{k_2}}{\omega_k}\right)^y - \left(\frac{\omega_{k_3}}{\omega_k}\right)^y\right) \times$$
$$\delta(\omega + \omega_1 - \omega_2 - \omega_3) d\omega_1 d\omega_2 d\omega_3$$

where

$$y = -\gamma + 3x - 3$$

Being rewritten in a form (3), kinetic equation explicitly exibits four stationary scaling solutions [1]. One of them corresponds to $y = 1$ and defines the final energy flux P from long to short scales. Rewriting integral (3) in dimensionless form on can get the expression for a collision integral on scaling spectrum (2):

$$\frac{\partial n_k}{\partial t} = \left(\frac{A}{\alpha}\right)^3 \frac{\omega_k^{-y-\frac{d}{\alpha}}}{C^{2\frac{d+\beta}{\alpha}}} I(y)$$

where

$$I(y) = \int_\Omega S\left(1, \xi_1^{\frac{1}{\alpha}}, \xi_2^{\frac{1}{\alpha}}, \xi_3^{\frac{1}{\alpha}}\right) (\xi_1 \xi_2 \xi_3)^{-x} (1 + \xi_1^x - \xi_2^x - \xi_3^x)$$
$$(1 + \xi_1^y - \xi_2^y - \xi_3^y) \delta(1 + \xi_1 - \xi_2 - \xi_3) d\omega_1 d\omega_2 d\omega_3$$

Here $\xi_i = (\frac{k_i}{k})^\alpha$ are dimensionless frequencies, $i = \overline{1,3}$.

Domain of integration Ω is defined from the condition that four vectors $\vec{k}$,$\vec{k_1}$,$\vec{k_2}$,$\vec{k_3}$ can form a fourangle on manifold

$$\omega + \omega_1 = \omega_2 + \omega_3$$

In other words domain Ω is defined from the conditions

$$\begin{cases} k_1 + k_2 + k_3 > k \\ \omega + \omega_1 = \omega_2 + \omega_3 \\ \omega_k = C(\nu) k^{\alpha} \\ \omega_2 < \omega, \ \omega_3 < \omega \end{cases}$$

Rewriting last conditions in dimensionless units, we get:

$$\Omega : \left\{ \ (\xi_2 + \xi_3 - 1)^{\frac{1}{\alpha}} + \xi_2^{\frac{1}{\alpha}} + \xi_3^{\frac{1}{\alpha}} > 1, \ \xi_2 < 1, \ \xi_3 < 1 \ \right\}$$

This domain of integration Ω is shown on Fig.3.

2 Energy flux and Kolmogorov constant

The energy flux P from the pumping region of long scales to the damping region of short scales at intermediate "transparency" region of frequencies ω_k is defined by the equation

$$\frac{\partial \epsilon_\omega}{\partial t} + \frac{\partial P}{\partial \omega} = 0 \tag{4}$$

where N_ω and $\epsilon_\omega = \omega N_\omega$ are the density of waves and the density of energy in the frequency space, correspondingly.

The number of waves in frequency space N_ω is connected with the number of waves in wavenumbers space $n_{\vec{k}}$ as

$$\int N_\omega d\omega = \int n_{\vec{k}} d\vec{k}$$

and, hence, in isotropical case

$$N_\omega = 4\pi n_k k^{d-1} \frac{dk}{d\omega} = 4\pi \frac{\omega^{\frac{d}{\alpha}-1}}{\alpha C^{\frac{d}{\alpha}}} n_k$$

The energy flux, according to (4) is

$$P = -\int \omega \dot{N}_\omega d\omega == -4\pi \frac{A^3}{\alpha^4 C^{\frac{2\beta+3d}{\alpha}}} \omega^{-y+1} \frac{I(y)}{-y+1} \tag{5}$$

For $y = 1$ (Kolmogorov spectrum) the expression $\frac{I(y)}{-y+1}$ contains indeterminancy $\frac{0}{0}$ (see Sec...) and can be regularized using L'Hopital rule:

$$P = 4\pi \frac{A^3}{\alpha^4 C^{\frac{2\beta+3d}{\alpha}}} \frac{\partial I(y)}{\partial y}$$

Spectral density of energy integrated over angles in $\vec{K}$-space is (see *Appendix A*):

$$I_k = 4\pi\epsilon_k = 4\pi R^2 B(\nu_0) k^{-\frac{5}{3}}$$

Comparing with definition of the Kolmogorov constant

$$I_k = C_{kolm} P^{\frac{2}{3}} k^{-\frac{5}{3}}$$

we have

$$C_{kolm} = \frac{4\pi R^2 B(\nu_0)}{P^{\frac{2}{3}}}$$

Plugging in here the expression (5) for the energy flux P we have

$$C_{kolm} = \frac{(4\pi)^{\frac{1}{3}} \alpha^{\frac{8}{3}} C^{\frac{2}{3}\frac{2\beta+3d}{\alpha} - 2x} B(\nu_0)}{\left(\frac{\partial I(y)}{\partial y}\right)^{\frac{2}{3}}} \tag{6}$$

For the Kolmogorov spectrum we have $y = y_0 = 1$. Remembering that for scaling spectrum we have

$$\begin{aligned} y &= -\gamma + 3x - 3 = 1 \\ \alpha &= \beta - \alpha x + d \\ \gamma &= \frac{2\beta + 3d}{\alpha} - 4 \\ \nu &= x\alpha \end{aligned}$$

we obtain the values of x, α and ν corresponding to Kolmogorov solution

$$x_0 = \frac{13}{2} \tag{7}$$

$$\alpha_0 = \frac{2}{3} \tag{8}$$

$$\nu_0 = \frac{13}{3} \tag{9}$$

It is easy to see now that power of C for these values of x and α in the expression (6) equal 0, i.e. C droppes out from the final expression for the Kolmogorov constant:

$$C_{kolm} = \frac{(4\pi)^{\frac{1}{3}} \alpha_0^{\frac{8}{3}} B(\nu_0)}{\left(\frac{\partial I(y)}{\partial y}\Big|_{y_0=1} \right)^{\frac{2}{3}}} \tag{10}$$

where

$$\frac{\partial I}{\partial y}\Big|_{y_0=1} = 2 \int_\Omega S\left(1, \xi_1^{\frac{1}{\alpha_0}}, \xi_2^{\frac{1}{\alpha_0}}, \xi_3^{\frac{1}{\alpha_0}}\right) (\xi_1\xi_2\xi_3)^{-x_0} \left(1 + \xi_1^{x_0} - \xi_2^{x_0} - \xi_3^{x_0}\right)$$
$$(\xi_1 \ln \xi_1 - \xi_2 \ln \xi_2 - \xi_3 \ln \xi_3)\, \delta\,(1 + \xi_1 - \xi_2 - \xi_3)\, d\xi_1 d\xi_2 d\xi_3$$

Using the expression (13) for $S\left(1, \xi_1^{\frac{1}{\alpha_0}}, \xi_2^{\frac{1}{\alpha_0}}, \xi_3^{\frac{1}{\alpha_0}}\right)$ from *Appendix B* we finally get

$$\frac{\partial I}{\partial y}\Big|_{y_0=1} = \frac{1}{2^5(2\pi)^3} \int_\Omega R\left(1, \xi_1^{\frac{3}{2}}, \xi_2^{\frac{3}{2}}, \xi_3^{\frac{3}{2}}\right) \left((\xi_1\xi_2\xi_3)^{-\frac{9}{2}}\right) \times \tag{11}$$
$$\left(1 + \xi_1^{\frac{13}{2}} - \xi_2^{\frac{13}{2}} - \xi_3^{\frac{13}{2}}\right) (\xi_1 \ln \xi_1 - \xi_2 \ln \xi_2 - \xi_3 \ln \xi_3) \times$$
$$\delta\,(1 + \xi_1 - \xi_2 - \xi_3)\, d\xi_1 d\xi_2 d\xi_3$$

Expressions (10), (11) and (12) present final formulae for a calculation of Kolmogorov constant.

Taking the integral (11) numerically we get

$$C_{kolm} = 4.7$$

3 Appendix A

In the *Appendix A* we are going to calculate the spectral density of energy ϵ_k on Kolmogorov solution (2), $\nu = \nu_0 = \frac{13}{3}$.

The spectral component of velocity

$$\vec{v_k} = \int \vec{\phi}_{\vec{k_1}\vec{k_2}} \delta(\vec{k} - \vec{k_1} + \vec{k_2}) a^*_{\vec{k_1}} a_{\vec{k_2}} d\vec{k_1} d\vec{k_2}$$

hence

$$\epsilon_{\vec{k}} = \frac{1}{2} < \vec{v}^*_{\vec{k}} \vec{v}_{\vec{k'}} >= \frac{1}{2} \int \vec{\phi}^2_{\vec{k_1}\vec{k_2}} n_{\vec{k_1}} n_{\vec{k_2}} \delta(\vec{k} - \vec{k_1} + \vec{k_2}) d\vec{k_1} d\vec{k_2}$$

where

$$< a_{\vec{k}} a^*_{\vec{k'}} >= n_{\vec{k}} \delta(\vec{k} - \vec{k'})$$

Here brackets <> mean an averaging over an ensemble.

It's easy to see that

$$\vec{\phi}_{\vec{k_1}\vec{k_2}} = -\frac{1}{(2\pi)^{\frac{3}{2}}} \frac{\left[\vec{k_1} - \vec{k_2}, \left[\vec{k_1}, \vec{k_2}\right]\right]}{|\vec{k_1} - \vec{k_2}|^2} = \frac{1}{2} \frac{1}{(2\pi)^{\frac{3}{2}}} \left[\vec{k_1} + \vec{k_2} - \frac{\left(\vec{k}, \vec{k_1} + \vec{k_2}\right)}{k^2} \vec{k}\right]$$

where

$$\vec{k} = \vec{k_1} - \vec{k_2}$$

and

$$\vec{\phi}^2_{\vec{k_1}\vec{k_2}} = \frac{1}{4} \frac{1}{(2\pi)^3} \left[2(k_1^2 + k_2^2) - k^2 - \frac{(k_1^2 - k_2^2)^2}{k^2}\right]$$

On the scaling solution (2) we have

$$\epsilon_{\vec{k}} = \frac{1}{8} \frac{R^2}{(2\pi)^3} \int \left[2(k_1^2 + k_2^2) - k^2 - \frac{(k_1^2 - k_2^2)^2}{k^2}\right] (k_1 k_2)^{-\nu} \delta(\vec{k} - \vec{k_1} + \vec{k_2}) d\vec{k_1} d\vec{k_2}$$

Now we will average $\epsilon_{\vec{k}}$ over the angles in $\vec{K}$-space This procedure is reduced to the averaging of δ-function over these angles. It is easy to see that

$$< \delta(\vec{k_1} - \vec{k_2} + \vec{k_3}) >= \frac{2\pi}{k_1 k_2 k_3}$$

where the brackets $<>$ mean an averaging over angles in $\vec{K}$-space and we have

$$\epsilon_k = R^2 B(\nu) k^{5-2\nu}$$

where

$$B(\nu) = \frac{1}{(2\pi)^2} \int_\Delta \left[2(u^2+v^2) - 1 - (u^2+v^2)^2\right] (uv)^{1-\nu}\, dudv$$

Here the integration is taken over half-infinite strip $\Delta : \{v < u+1, v > u-1, v > u+1\}$ (see Fig.4a). For the numerical integration purposes it's convenient to turn the system of coordinates by $\frac{\pi}{2}$ Coordinate transformation

$$u = \frac{x-y}{\sqrt{2}} + \frac{1}{2}$$
$$v = \frac{x+y}{\sqrt{2}} + \frac{1}{2}$$

maps the domain Δ into domain $\Delta' : \{-\frac{1}{\sqrt{2}} < y < \frac{1}{\sqrt{2}}, x > 0\}$ with Jakobian $\frac{\partial(u,v)}{\partial(x,y)} = 1$

This mapping gives for Kolmogorov spectrum ($\nu = 1$)

$$B\left(\frac{13}{3}\right) = \frac{2^{\frac{7}{3}}}{(2\pi)^2} \int_{\Delta''} \frac{x(x+\sqrt{2})(1-2y^2)}{(x-y+\frac{1}{\sqrt{2}})^{\frac{10}{3}}(x+y+\frac{1}{\sqrt{2}})^{\frac{10}{3}}} dxdy$$

Here the integration is carried over one half of domain Δ', i.e. $\Delta'' : \{0 < x < \infty, 0 < y < \frac{1}{\sqrt{2}}\}$ (see Fig.4b).

From numerical integration we get

$$B\left(\frac{13}{3}\right) = 0.047 \tag{12}$$

4 Appendix B

In the *Appendix B* we are calculating an analitical expression for the average of the matrix element over eight angles in $\vec{K}$-space.

The configuration of four vectors $\vec{k}$, $\vec{k_1}$, $\vec{k_2}$, $\vec{k_3}$ is invariant with respect to rotations around the vector $\vec{p} = \vec{k_1} - \vec{k_3} = \vec{k_4} - \vec{k_2}$ (see Fig.1) as a whole. This fact allows to reduce the integration over four angles ϕ to two angles ϕ.

The angles θ can be expressed through the absolute values of vectors $k = |\vec{k}|$, $k_1 = |\vec{k_1}|$, $k_2 = |\vec{k_2}|$, $k_3 = |\vec{k_3}|$, $p = |\vec{k} - \vec{k_2}| = |\vec{k_3} - \vec{k_2}|$ as

$$
\begin{aligned}
\cos\theta &= \frac{k^2 + p^2 - k_2^2}{2pk} \\
\cos\theta_1 &= \frac{k_3^2 - p^2 - k_1^2}{2pk_1} \\
\cos\theta_2 &= \frac{k^2 - p^2 - k_2^2}{2pk_2} \\
\cos\theta_3 &= \frac{k_3^2 + p^2 - k_1^2}{2pk_3}
\end{aligned}
$$

Using these properties we get the expression for function $S(1, \xi_1^{\frac{1}{\alpha}}, \xi_2^{\frac{1}{\alpha}}, \xi_3^{\frac{1}{\alpha}})$ which we will need for our purposes:

$$
S(1, \xi_1^{\frac{1}{\alpha}}, \xi_2^{\frac{1}{\alpha}}, \xi_3^{\frac{1}{\alpha}}) = \frac{1}{2^6(2\pi)^3} R(\xi_1^{\frac{1}{\alpha}}, \xi_2^{\frac{1}{\alpha}}, \xi_3^{\frac{1}{\alpha}})(\xi_1\xi_2\xi_3)^{\frac{d-1}{\alpha}-1} \tag{13}
$$

where function R is the following:

$$
\begin{aligned}
&R(\kappa_1, \kappa_2, \kappa_3) = \\
&2\int_{1-\kappa_2}^{\kappa_1+\kappa_3} \frac{((\kappa_2-1)^2-\kappa^2)(\kappa^2-(1+\kappa_2)^2)((\kappa_3-\kappa_1)^2-\kappa^2)(\kappa^2-(\kappa_1+\kappa_3)^2)}{\kappa^4} d\kappa \\
&+\int_{1-\kappa_3}^{\kappa_1+\kappa_2} \frac{((\kappa_3-1)^2-\kappa^2)(\kappa^2-(1+\kappa_3)^2)((\kappa_2-\kappa_1)^2-\kappa^2)(\kappa^2-(\kappa_1+\kappa_2)^2)}{\kappa^4} d\kappa \\
&-4\left[(C_0 - C_1 - \frac{\kappa^2}{3})\sqrt{C_2\kappa^2 - C_3} + C_4 ArcTan\left(\sqrt{\frac{2\kappa^2}{3C_1} - 1}\right) - 2C_2\kappa\right]\Bigg|_{\kappa=1-\kappa_2}^{\kappa=\kappa_1+\kappa_3}
\end{aligned}
$$

where

$$
\begin{aligned}
C_0 &= 1+\xi_1^3+\xi_2^3+\xi_3^3 \\
C_1 &= 2\frac{(1-\xi_2)^2(\xi_2+\xi_3)(\xi_2^2\xi_3^2+\xi_1\xi_2\xi_3+\xi_3^2)}{(\xi_1^2+\xi_1\xi_2+\xi_2^2)(\xi_3^2+\xi_3+1)} \\
C_2 &= (1-\xi_3)^2(\xi_1^2+\xi_1\xi_2+\xi_2^2)(\xi_3^3+\xi_3+1) \\
C_3 &= 3(1-\xi_3)^2(1-\xi_2)^2(\xi_2+\xi_3)(\xi_2^2\xi_3^2+\xi_1\xi_2\xi_3+\xi_1^2) \\
C_4 &= -\frac{(1-\xi_2)(1-\xi_3)(\xi_1^2+\xi_1\xi_2+\xi_2^2)(\xi_3^2+\xi_3+1)(\xi_2^2+\xi_2+1)(\xi_3^2+\xi_1\xi_3+\xi_1^2)}{\sqrt{3(\xi_2+\xi_3)(\xi_2^2\xi_3^2+\xi_1\xi_2\xi_3+\xi_1^2)}}
\end{aligned}
$$

Here $\kappa_i = \frac{k_i}{k}$ and $\xi_i = \kappa_i^\alpha$ are dimensionless wavenumbers and frequencies, correspondingly, $i = \overline{1,3}$.

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[1] S.Dyachenko, A.C.Newell, A.Pushkarev, V.R.Zakharov, "Optical turbulence: weak turbulence, condensates and collapsing filaments in the nonlinear Schroedinger equation", *Physica D*, 57, pp.96-160

ON THE TURBULENCE OF CAPILLARY WAVES

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Abstract

An ensemble of weakly interacting capillary waves on the surface of ideal fluid forms a kind of wave turbulence. Similary to the case of "classical" turbulence in incompressible three-dimensional fluid, the main physical process in this turbulence is the energy transport in K-space from large to small scales.

According to the weak-turbulent theory, the spatial spectrum of elevations $I_k =< |\eta_k| >^2$ in this turbulence is the Kolmogorov-type spectrum $I_k \simeq k^{-\frac{19}{4}}$. So far this result was not confirmed neither experimentally nor numerically.

We developed a numerical scheme for direct simulation of the surface of ideal fluid based on the expansion of the Hamiltonian of the surface up to terms of fourth order and observed the predicted Kolmogorov spectrum.

Introduction

An ensemble of weakly interacting waves in a dispersive medium can be described statistically even being very far from the state of thermodynamic equilibrium. Due to small value of of nonlinearity the infinite system of equations for corellation function in this case can be truncated by a consistent way and reduced to one kinetic equation for "wave numbers" (or wave action)

$$\frac{\partial n_{\vec{k}}}{\partial t} + 2\gamma_{\vec{k}} n_{\vec{k}} = st(n) \tag{1}$$

(see, for instanse, [2]). Here $\gamma_{\vec{k}}$ is the wave damping (or the wave pumping if $\gamma_{\vec{k}} < 0$), $st(n)$ is the "collision term" corresponding to wave equation.

The collision term describes "cascades"-transport of wave energy in K-space to small scales region (direct cascade) and to the large scale region (inverse cascade). The last one existy only if the total number of waves $N = \int n_{\vec{k}} d\vec{k}$ is the integral of motion.

The equation

$$st(n_{\vec{k}}) = 0$$

besides trivial thermodynamic solution has Kolmogorov-type solutions describing cascades. In a medium without a characteristic length they are powerlike functions

$$n_{\vec{k}} \simeq k^{-\alpha} \tag{2}$$

The theory of weak-turbulent Kolmogorov spectra now is far advanced. But direct experimental confirmation of these spectra are very poor. One can consider more or less well confirmed existence of the Komogorov spectrum for the direct cascade of gravitational wave on the surface of incompressible deep fluid

$$I_\omega \simeq \alpha \frac{gv}{\omega^4}$$

(Here $I(\omega)$-spectral density of surface elevations, ω- wave frequency, g is gravity acceleration, v-wind velocity, α - dimensionless constant). This

spectrum was theoretically derived by Zakharov and Filonenko [4] and experimentally observed by Toba [5].

Another way to check the weak-turbulent thory is numerical simulation. Some valuable results were obtained by numerical solution of kinetic equation (1) [7], [8]. But the kinetic equation (1) is itself a subject for careful examination. Its derivation assumes that the phases of all interacting waves are random and are in the state of chaotic motion. The validity of this assumption is no clear apriory.

The right way to check the weak-turbulent theory and its prediction is numerical simulation "from the first principles", i.e. direct solution of the nonlinear dynamic equation governing propagation and interaction of the waves.

In real cases these equations are of two or three spatial dimensions, and its numerical solution is not a simple problem. It was done so far for $2D$ Nonlinear Schroedinger Equation [10], but in this particular case Kolmogorov spectra don't exist.

In this paper we present results of numerical simulation of capillary waves on the surface of the incompressible infinitely deep fluid. In this case only direct cascade of energy takes place. Corresponding Kolmogorov spectrum for the surface elevation has a form $I_k \simeq k^{\frac{-17}{4}}$. We will show that this theoretical prediction is confirmed by direct numerical simulation with good accuracy. The developed numerical approach can be used for solution of a wide class of problem pertained to interaction of surface waves and - more generally - other types of waves in nonlinear media.

Theoretical background

We study the potential flow of ideal incompressible deep fluid with the free surface. Let $\eta(\vec{r},t)$, $\vec{r} = (x,y)$ is the shape of the surface, $\psi(\vec{r},t)$ is the velocity potential $\Phi = \Phi(\vec{r},z)$, $\vec{v} = \nabla\Phi$, $\Delta\Phi = 0$, evaluated on the free surface: $\Psi(\vec{r},t) = \Phi(\eta(\vec{r},t),\vec{r},t)$. It is known [9] that under these assumption the fluid is a Hamiltonian system:

$$\frac{\partial\eta}{\partial t} = \frac{\delta H}{\delta\Psi} \tag{3}$$

$$\frac{\partial \Psi}{\partial t} = -\frac{\delta H}{\delta \eta} \tag{4}$$

Here H is the total energy of the fluid consisting of the kinetic and the potential components

$$H = H_{pot} + H_{kin} \tag{5}$$

where

$$H_{pot} = \sigma \int [(1 + (\nabla \eta)^2)^{\frac{1}{2}} - 1] d\vec{r}$$

$$H_{kin} = \frac{1}{2} \int d\vec{r}_\perp \int_{-\infty}^{\eta} dz (\nabla \Phi)^2$$

Here σ is a coefficient of surface tension.

Direct numerical simulation of the system (3), (4) provides solution of the boundary problem for the Laplace equation on each step in time. In full $3D$ case it is enormously hard problem. To solve the problem one can use an expansion in powers of nonlinearity. For Fourier transforms this expansion up to the quadratic terms has a form:

$$\begin{aligned}
H &= H_0 + H_1 + H_2 + \ldots \\
H_0 &= \frac{1}{2} \int \left[|k| |\psi_{\vec{k}}|^2 + \sigma |k|^2) |\eta_{\vec{k}}|^2 \right] d\vec{k} \\
H_1 &= -\frac{1}{2 \times 2\pi} \int L_{\vec{k_1} \vec{k_2}} \psi_{\vec{k_1}} \psi_{\vec{k_2}} \eta_{\vec{k_3}} \delta(\vec{k}_1 + \vec{k}_2 + \vec{k}_3) d\vec{k}_1 d\vec{k}_2 d\vec{k}_3 \\
H_2 &= \frac{1}{4(2\pi)^2} \int M_{\vec{k}\vec{k_1}\vec{k_2}\vec{k_3}} \Psi_{\vec{k}} \Psi_{\vec{k_1}} \eta_{\vec{k_2}} \eta_{\vec{k_3}} \delta(\vec{k} + \vec{k}_1 + \vec{k}_2 + \vec{k}_3) d\vec{k} d\vec{k}_1 d\vec{k}_2 d\vec{k}_3 \\
L_{\vec{k_1} \vec{k_2}} &= \vec{k_1} \vec{k_2} + |\vec{k_1}| |\vec{k_2}| \\
M_{\vec{k_1} \vec{k_2} \vec{k_3} \vec{k_4}} &= |\vec{k_1}| |\vec{k_2}| [\frac{1}{2} (|\vec{k_1} + \vec{k_3}| + |\vec{k_1} + \vec{k_4}| + |\vec{k_2} + \vec{k_3}| + |\vec{k_2} + \vec{k_4}|) \\
&\quad - |\vec{k_1}| - |\vec{k_2}|]
\end{aligned}$$

Corresponding dynamic equations are

$$\frac{\partial \eta_{\vec{r}}}{\partial t} = \left[|\hat{k}| \psi \right]_{\vec{r}} - div(\eta \nabla \psi) - |\hat{k}| \left[\left[|\hat{k}| \psi \right]_{\vec{r}} \times \eta_{\vec{r}} \right]_{\vec{r}} + |\hat{k}| \left[|\hat{k}| \left[\left[|\hat{k}| \psi \right]_{\vec{r}} \times \eta_{\vec{r}} \right]_{\vec{r}} \times \eta_{\vec{r}} \right]_{\vec{r}}$$

$$+\frac{1}{2}\Delta_{\vec r}\left[\left[|\hat k|\psi\right]_{\vec r}\times\eta^2_{\vec r}\right]_{\vec r}+\frac{1}{2}|\hat k|\left[\Delta_{\vec r}\times\eta^2_{\vec r}\right] \tag{6}$$

$$\begin{aligned}\frac{\partial\psi_{\vec r}}{\partial t} &= \sigma\Delta_{\vec r}\eta_{\vec r}+\frac{1}{2}\left[-(\nabla\psi)^2+\left[|\hat k|\psi\right]^2_{\vec r}\right]-|\hat k|\left[\left[|\hat k|\psi\right]_{\vec r}\times\eta_{\vec r}\right]_{\vec r}\times\left[|\hat k|\psi\right]_{\vec r}\\ &\quad -\Delta\psi\times\left[|\hat k|\psi\right]_{\vec r}\times\eta_{\vec r}+D_{\vec r}+F_{\vec r}\end{aligned} \tag{7}$$

We added to the equation (7) a phenomenological damping term $D_{\vec r}$ and the external force $F_{\vec r}(t)$.

In the linear approximation the equations (6) and (7) describe capillary waves with the dispersion relation

$$\omega_k=(\sigma k^3)^{\frac{1}{2}}$$

One can introduce normal amplitudes

$$a_{\vec k}=\sqrt{\frac{\sigma k^2}{2\omega_k}}\eta_{\vec k}-i\sqrt{\frac{k}{2\omega_k}}\psi_{\vec k}$$

According to the weak-turbulent theory the pair corellation function

$$<a_{\vec k}a^*_{\vec k'}>=n_{\vec k}\delta(\vec k-\vec k')$$

satisfies the kinetic equation (1), where

$$\begin{aligned}st(n) &= \int\left[R_{\vec k\vec k_1\vec k_2}-R_{\vec k_1\vec k\vec k_2}-R_{\vec k_2\vec k\vec k_1}\right]d\vec k d\vec k_1\\ R_{\vec k\vec k_1\vec k_2} &= 4\pi|V_{\vec k\vec k_1\vec k_2}|^2\delta(\vec k-\vec k_1-\vec k_2)\delta(\omega_{\vec k}-\omega_{\vec k_1}-\omega_{\vec k_2})\\ &\quad\left[n_{\vec k_1}n_{\vec k_2}-n_{\vec k}n_{\vec k_1}-n_{\vec k}n_{\vec k_2}\right]\\ V_{\vec k\vec k_1\vec k_2} &= \frac{1}{2^{\frac{5}{2}}2\pi}(\omega_{k_1}\omega_{k_2}\omega_{k_3})^{\frac{1}{2}}\left[\frac{k^2L_{\vec k_1,\vec k_2}}{k_1k_2}-\frac{k_2^2L_{\vec k,-\vec k_1}}{kk_1}-\frac{k_1^2L_{\vec k,-\vec k_2}}{kk_2}\right]\end{aligned}$$

In isotropic medium containing no characteristic length the dispersion relation is a power-like function

$$\omega_k \simeq k^{\alpha}$$

as far as $V_{\vec{k}\vec{k}_1\vec{k}_2}$ is a homogeneous function

$$V_{\epsilon\vec{k},\epsilon\vec{k}_1,\epsilon\vec{k}_2} = \epsilon^{\beta} V_{\vec{k}\vec{k}_1\vec{k}_2}$$

In this case the equation

$$st(n) = 0$$

has exact powerlike solution

$$n_k = C\frac{P^{\frac{1}{2}}}{k^{\beta+d}}$$

(d is dimension of space), which is Kolmogorov-type spectrum describing constant flux of energy in K-space from large to small scales. P is the value of the energy flux, C is an absolute constant. For capillary waves $\alpha = \frac{3}{2}$, $\beta = \frac{9}{4}$, $d = 2$. Hence

$$n_k = C\frac{P^{\frac{1}{2}}}{k^{\frac{17}{4}}} \tag{8}$$

For the corellation function of elevation one get

$$I_k =< |\eta_k|^2 >= \frac{\omega_k}{\sigma k^2} = \frac{C\sigma^{-\frac{1}{4}}p^{\frac{1}{2}}}{k^{\frac{19}{4}}}$$

This result was obtained first by Zakharov and Filonenko [4]. The solution (8) is linearly stable in framework of the kinetic equation (1) (see [11]).

Numerical simulation

We realized numerical simulation of the system (6), (7). In spite of the fact that the matrix element of the kinetic equation $V_{\vec{k}\vec{k}_1\vec{k}_2}$ is expressed only through coefficients of the Hamiltonian H_0, H_1 we prefer to keep the next term H_2 in the expansion of the Hamiltonian. The reason for that is the following - it can be shown that the dynamical system generated by the Hamiltonian $H_0 + H_1$ becomes ill-posed at very low levels of nonlinearity. Meanwhile, including into consideration the next term of expansion improves the situation essentially (details of the consideration will be published separately). Moreover, the developed scheme after a minor modification can be used for numerical simulation of the gravitational waves.

The equations (6) and (7) are not differential in the X-space. Besides taking derivatives they include taking of the operator $|k|$ ($(-\Delta)^{\frac{1}{2}}$ in X-space).

So, the system can be reduced to the set of 6 PDE for the variables interconnected by the consequent application of the operator $(-\Delta)^{\frac{1}{2}}$. It allows to apply for solution of the system (6) and (7) the spectral code using the Fast Fourier Transform on each step of time. Omitting the details of numerical scheme, we reproduce now only the final result of calculation.

For numerical integration of the equations (6)-(7) we used the functions F and D defined at Fourier space through the following relations :

$$
\begin{aligned}
F_{\vec{k}} &= f_{\vec{k}} e^{i\Omega_{\vec{k}} t} \\
D_{\vec{k}} &= \gamma_{\vec{k}} \Psi_{\vec{k}} \\
\Omega_{\vec{k}} &= \omega_{\vec{k}}(1 + R(t)) \\
\gamma_{\vec{k}} &= \begin{cases} -(|\vec{k}| - |\vec{k_0}|)^2 & \text{if } k > k_0 \\ 0 & \text{if } 0 \le k \le k_0 \end{cases}
\end{aligned}
$$

Pumping force F_k is "almost" in resonance with local linear frequency ω_k of the corresponding Fourier-garmonics, i.e. frequency Ω_k slightly fluctuate around exact value of ω_k due to small random in time addition $R(t)$. The form of pumping amplitude was chosen to be axially-symmetric $f_k = f_0 e^{\left(\frac{|\vec{k}| - |\vec{k_1}|}{k_2}\right)^4}$.

Value k_0 defines starting point of "hyperviscosity" we used in our experiments to provide wide enough intertial interval. Calculations were carried out on the grid 256×256.

System was driven by forcing F localized at small wavenumbers. After a while, we observed formation of the stationary spectrum of waves carrying constant in time energy flux to the high $\vec{k}$ due to nonlinear interaction of waves. Observed spectrum was characterized by angular isotropy and being averaged over the angles in K-space apeared to be in a good agreement for the inertial interval with the spectrum predicted by the weak-turbulent theory (8).

This stationary spectra were obtained for and nonlinearity levels $\frac{H_l}{H_{nl}} \simeq 0.05$ (here H_l and H_n are linear and nonlinear part of the energy).

Conclusions

Summarizing the results we can conclude that the direct numerical simulation of the dynamic nonlinear equation confirms an existence and important role of the weak-turbulent Kolmogorov spectra at least in the case of capillary waves. Indirectly this result confirms the validity of the kinetic equation for a description of the wave turbulence. We hope that the developed effective approach will allow us to study numerically other types of wave turbulence, first of all - the behavior of a system of wind-driven gravitational waves on the sea-surface.

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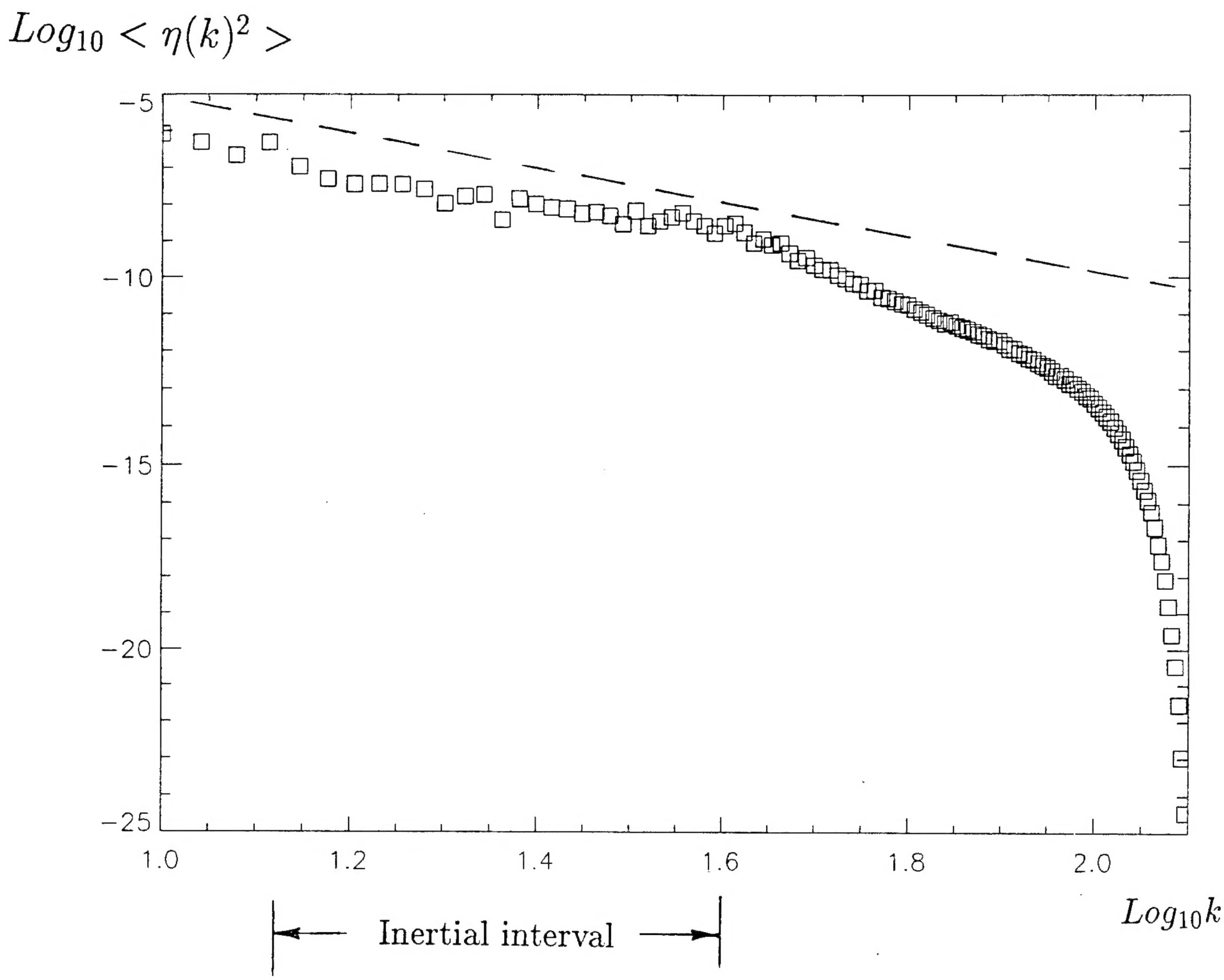

Dashed line corresponds to the spectrum
$< \eta(k)^2 > \sim k^{-\frac{19}{4}}$ predicted by weak-turbulent theory